AD-A150 772 AN EVALUATION OF FINITE ELEMENT MODELS FOR SOIL CONSOLIDATION(U) OHIO STATE UNIV RESEARCH FOUNDATION COLUMBUS R S SANDHU ET AL. APR 84 OSURF-715107-84-2 1/1

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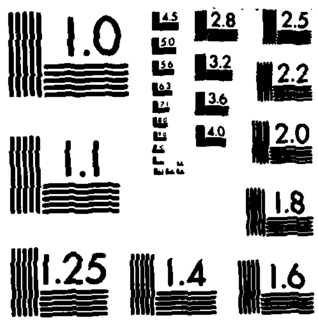
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RF Project 763420/715927
Annual Report

AN EVALUATION OF FINITE ELEMENT MODELS FOR SOIL CONSOLIDATION

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Department of Civil Engineering

DEPARTMENT OF THE AIR FORCE
Air Force Office of Scientific Research
Bolling Air Force Base, D.C. 20332

Grant No. AFOSR-83-0055

April, 1984

The Ohio State University
Research Foundation
1314 Kinnear Road
Columbus, Ohio 43212

85 02 13 058

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED

1b. RESTRICTIVE MARKINGS:

2a. SECURITY CLASSIFICATION AUTHORITY:

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE:

3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for Public Release Distribution Unlimited.

4. PERFORMING ORGANIZATION REPORT NUMBER(S): OSURF - 715107 - 84 - 2

5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 85-0079

6a. NAME OF PERFORMING ORGANIZATION: The Ohio State Research Foundation

6b. OFFICE SYMBOL (If applicable):

7a. NAME OF MONITORING ORGANIZATION:

6c. ADDRESS (City, State and ZIP Code): Department of Civil Engineering, 1314 Kinnear Road, Columbus, OH 43212

7b. ADDRESS (City, State and ZIP Code):

8a. NAME OF FUNDING/SPONSORING ORGANIZATION: Air Force Office of Scientific Research

8b. OFFICE SYMBOL (If applicable): AFOSR/NA

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: AFOSR - 83- 0055

8c. ADDRESS (City, State and ZIP Code): Bolling AFB, DC 20332-6448

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2307	C1	

11. TITLE (Include Security Classification): An Evaluation of Finite Element Models for Soil Consolidation (Unclassified)

12. PERSONAL AUTHOR(S): Baher L. Aboustit Ranbir S. Sandhu Soon J. Hong

13a. TYPE OF REPORT: INTERIM

13b. TIME COVERED: FROM ~~1 Feb 83~~ TO ~~31 Jan 84~~

14. DATE OF REPORT (Yr., Mo., Day): April, 1984

15. PAGE COUNT: 45

16. SUPPLEMENTARY NOTATION:

17. COSATI CODES

FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): Consolidation; Seepage; Finite Element Methods; Flow Through Deformable Porous Media

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Numerical performance of Ghaboussi's isoparametric bilinear quadrilateral element, for analysis of quasi-static flow of an incompressible fluid through a linear elastic saturated porous soil, is compared with that of Sandhu's composite element in which the displacement has biquadratic interpolation. Application of both procedures to solution of one-dimensional consolidation and plain strain consolidation of the half-space under a strip load shows that Ghaboussi and Wilson's procedure gives results almost identical to those from the higher order element but is significantly more economical to use.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED XX SAME AS RPT ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION: Unclassified

22a. NAME OF RESPONSIBLE INDIVIDUAL: Lt. Col. Lawrence D. Hokanson

22b. TELEPHONE NUMBER (Include Area Code): 202/767-4935

22c. OFFICE SYMBOL: AFOSR/NA

DD FORM 1473, 83 APR EDITION OF 1 JAN 73 IS OBSOLETE. Unclassified

SECURITY CLASSIFICATION OF THIS PAGE

OSURF-715107-84-2

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AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
Grant: AFOSR-83-0055

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Geotechnical Engineering Report No. 11

The Ohio State University Research Foundation
1314 Kinnear Road, Columbus, Ohio 43212

FOREWORD

The investigation reported herein is part of the research project at The Ohio State University, Columbus, Ohio supported by the Air Force Office of Scientific Research Grant 83-00-55. Lt. Col. John J. Allen was the Program Manager at the commencement of the project. Lt. Col. Lawrence D. Hokanson was the Program Manager from July 1, 1983. The research project was started Feburary 1, 1983 and is continuing. The present report documents part of the work done up to January 31, 1984. At The Ohio State University, the project is supervised by Dr. Ranbir S. Sandhu, Professor, Department of Civil Engineering. The computer program modification and analyses reported herein were started by Dr. Baher L. Aboustit, Post-doctoral Reseach Associate and completed by Mr. Soon-jo Hong, Graduate student, who also prepared the documentation on the Ohio State University Computer System. The Instruction and Research Computer Center at The Ohio State University provided the computational facilities.

ABSTRACT

Numerical performance of Ghaboussi and Wilson's isoparametric bilinear quadrilateral element, for analysis of quasi-static flow of an incompressible fluid through a linear elastic saturated porous soil, is compared with that of Sandhu's composite element in which the displacement has biquadratic interpolation. Application of both procedures to solution of Terzaghi's one-dimensional consolidation and Gibson's problem of plane strain consolidation of the half-space under a strip load shows that Ghaboussi and Wilson's procedure gives results almost identical to those from the higher order element but is significantly more economical to use.

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SECTION I

INTRODUCTION

Since the first application of the finite element method to analysis of seepage in elastic media (1,2)* considerable progress has been made in the theoretical formulation as well as computational procedures. Recent advances include variational formulations admitting limited smoothness of finite element bases (3,4), experimentation with several different spatial interpolation schemes and investigation of various temporal approximation methods (5-15). The finite element method has been applied to soils exhibiting secondary compression (8,11,13), nonlinear soil behavior (13,14-18), and to finite deformation (16,18). Developments in solution procedures include use of Laplace transforms (10,11,19), automatic selection of the time-step size (13) and the use of single variable formulations (20,21).

In spatial discretization, Sandhu (1) proposed that the order of terms appearing in a convolution product in the variational principle be the same. This produced the "composite" element in which the order of polynomial interpolation for displacements was higher than that for fluid pressures. The composite element first proposed by Sandhu (1,2), and used by Hwang (22) and others, was the "63" element with quadratic interpolation for displacements and linear interpolation for fluid pres-

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* Numerals in parentheses refer to corresponding items in SECTION VII REFERENCES

sures over triangular regions. Later, Sandhu (7,23) introduced the "84" element which had eight point biquadratic interpolation for displacements and was a four point isoparametric quadrilateral for fluid pressures. This element was also used by Runesson (13). Buchmaier (24) experimented with five, six and seven point quadrilaterals for fluid pressure as transition elements near loaded surfaces.

Several spatial interpolation schemes, besides the composite elements, have been tried by various investigators. Ghaboussi (5) used four point isoparametric quadrilaterals for both the fields. However, an additional incompatible mode was included in the displacement approximation. This element has the economy while the additional "local" mode gives it the character of a "higher order" scheme. Smith's (25) formulation was similar to Ghaboussi's except that no incompatible modes were used. Prevost (9) proposed cautious use of "reduced integration" in conjuction with Smith's "44" element. Yokoo (26), Booker (19), and Vermeer (27) used triangular elements with linear interpolation for both the displacement and the fluid pressure fields. Other interpolation schemes, based on use of quadrilaterals built up from linear four or five point triangles were tried by Sandhu (8).

All investigators have generally reported success with whatever scheme they used. Comparative studies of different elements are rare. Some comparisons of nemerical performances were attemped by Sandhu (7,8). In evaluating various candidate schemes, Sandhu proposed that an acceptable method meet the following requirements in addition to efficiency and accuracy.

i. The interpolation scheme must conform with the assumptions regarding continuity and differentiability used in setting up the governing variational formulation.

ii. It should be possible to generate the "undrained" solution, i.e. the state of fluid pressures and displacement at time $t = 0+$.

iii. For sufficiently small time steps, the scheme should be insensitive to the choice of the time-step size.

Elements "63" and "84" satisfy these requirements. However, the composite elements are too expensive to be used in large problems. This has discouraged extension of the analysis to three-dimensions, and to nonlinear and dynamic problems. The 4-4 element is more economical but was found by Sandhu (8) to have oscillatory errors. The "63" element has been widely used because it was the first one to be introduced and gave satisfactory results in most cases. However, the 8-4 element gives results almost identical to those from the "63" element but is more economical as it requires fewer nodal points and has less band-width. Ghaboussi's element (5) was not included in this comparison.

The purpose of the present investigation was to implement Ghaboussi's formulation (we refer to it as the 6-4 element because of the additional internal modes in the displacement representation) and to compare its performance with that of the 8-4 element. Both the elements were used to solve Terzaghi's problem of one-dimensional consolidation and Gibson's problem of a half space under a strip load. To simulate consolidation of the half space by a finite domain model, an approximation to the conditions at infinity is required. Three alternative assumptions regarding the "cut off" boundary were considered.

SECTION II summarizes the governing field equations following Ghaboussi (5). The finite element development is given in SECTION III. SECTION IV describes the example problems. Results of the analysis are given in SECTION V.

SECTION II

EQUATIONS GOVERNING LINEAR ELASTIC SOIL CONSOLIDATION

Assuming pore water to be compressible, the equations of force equilibrium of elementary volumes and mass continuity, over the spatial region of interest R, may be written in standard indicial notation as, [Appendix A],

$$[E_{klij}u_{k,l}]_{,i} + \alpha\pi_{,j} + f_j = 0 \tag{1}$$

$$[K_{ij}(\pi_{,i} + \rho_2 f_i)]_{,j} + u_{j,j} = \frac{1}{M}\dot{\pi} \tag{2}$$

where u_i, f_i, E_{klij}, K_{ij}, denote the cartesian components, respectively, of the displacement vector, the body force vector per unit mass, the isothermal elasticity tensor and the permeability tensor. ρ is the mass density of the saturated soil and ρ_2 that of water. π is the pore water pressure, α is the solid compressibility and M is a measure of fluid compressibility. With these field equations we associate the following boundary conditions;

$$u_i = \hat{u}_i \qquad \text{on } S_{1i} \tag{3}$$

$$t_i = \tau_{ij}n_j = \hat{t}_i \qquad \text{on } S_{2i} \tag{4}$$

$$\pi = \hat{\pi} \qquad \text{on } S_3 \tag{5}$$

$$Q = q_i n_i = \hat{Q} \qquad \text{on } S_4 \tag{6}$$

Here, t_i, q_i are components of the traction and fluid flux vectors associated with surfaces embedded in the closure of R. τ_{ij} are components of the total stress tensor. S_{1i}, S_{2i} are complementary subsets of the boundary of the spatial region of interest and so are S_3, S_4. Even though the equations given above apply to compressible fluids, the applications reported herein assumed incompressible fluid i.e. $M \rightarrow \infty$. The initial conditions for the problem are

$$u_i(x_j, 0) = u_i(0) \quad \text{on } R$$

$$\pi(x_j, 0) = \pi(0) \quad \text{on } R$$

SECTION III

FINITE ELEMENT FORMULATION

Discretization of the governing function for the two-field formulation followed by application of the variational principle (5) leads to the following matrix equation.

$$\begin{bmatrix} K_{uu} & K_{pu} \\ K_{pu} & [-\alpha\Delta t K_{pp} - C_{pp}] \end{bmatrix} \begin{Bmatrix} u(t_1) \\ \pi(t_1) \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ -K_{pu} & [-(1-\alpha)\Delta t K_{pp} + C_{pp}] \end{bmatrix} \begin{Bmatrix} u(t_o) \\ \pi(t_o) \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \quad (7)$$

where (t_o, t_1) is the single time step of interest, and;

$\Delta t = t_1 - t_o$

$\{u(t_1)\}, \{u(t_o)\}$ = vectors of nodal point values of the components of the displacement at time t_1, t_o, respectively

$\{\pi(t_1)\}, \{\pi(t_o)\}$ = vectors of nodal point values of the pore water pressure at time t_1, t_o, respectively.

$\{P_1\}$ = the vector of nodal point loads including applied nodal loads, boundary tractions, body forces, initial stresses and effect of displacement constraints.

$\{P_2\}$ = the vector of nodal point fluxes including applied nodal fluxes, boundary fluxes, body force effects and effects of specified pore water pressures.

$[K_{uu}]$ = the spatial "stiffness matrix" for the elastic soil.

$[K_{pp}]$ = the spatial "flow matrix" for the compressible fluid and $\Delta t = 1$.

α = the coefficient characterizing single-step temporal discretization.

$[K_{pu}]$= the coupling matrix representing the influence of pore pressure in the force equilibrium equation.

$[K_{pu}]^T$= the coupling matrix representing the influence of soil volume change upon the nodal point fluid pressure.

$[C_{pp}]$= the spatial fluid compressibility matrix.

The matrix $[K_{uu}]$ depends upon the interpolation scheme for displacements and $[C_{pp}]$, $[K_{pp}]$ depend upon the interpolation scheme for the pore-water pressures. The coupling matrix $[K_{pu}]$ involves spatial interpolation for both the field variables. The temporal discretization for the single step scheme is reflected in the value of the coefficient α. For linear interpolation $\alpha = 0.5$.

Equation (7) includes the "natural" boundary conditions expressed by Equation (4) and (6). Equation (3) and (5) are satisfied by explicitly requiring $u_i = \hat{u}_i$ on S_{1i} and $\pi = \hat{\pi}$ on S_3. Development of the vectors and matrices appearing in Equation (7) is given in Appendix B. As the fluid is assumed to be incompressible in both the example problems, the matrix C_{pp} is zero. It is included in the theoretical discussion for completeness. The computer program can handle compressible as well as incompressible fluids.

SECTION IV

EXAMPLE PROBLEMS

An existing computer program, initially developed by Dr. J.K. Lee of the Department of Engineering Mechanics, The Ohio State University, was modified to include plane strain consolidation using both the 8-4 and the 6-4 elements. The code was used to solve Terzaghi's problem of one-dimensional consolidation and Gibson's problem of two-dimensional consolidation. For these problems, the theoretical solutions are known and, therefore, precise comparison was possible.

For Terzaghi's problem, the dimensions of the consolidating soil column and soil properties were the same as in Sandhu's example (23); i.e. soil depth h = 7, modulus of elasticity E = 6000, Poisson's ratio = 0.4, coefficient of permeability $K = 4 \times 10^{-6}$. Soil and water particles were assumed to be incompressible, i.e. $\alpha = 1$, $M \rightarrow \infty$ in Equation (7). Figure 1 illustrates the problem and Figure 2 illustrates the mesh used in the analysis. The mesh consisted of 9 elements with 20 nodes when using the 6-4 element and 48 nodes when using the 8-4 element. In temporal discretization, the coefficient α was given the value of 0.5. The following temporal partition was considered.

10 steps of Δt = 0.000001	over [0,0.00001]
10 steps of Δt = 0.01	over [0.00001,0.1]
10 steps of Δt = 0.1	over [0.1, 1.1]
10 steps of Δt = 10	over [1.1, 101.1]
10 steps of Δt = 100	over [101.1, 901.1]

where unit is second.

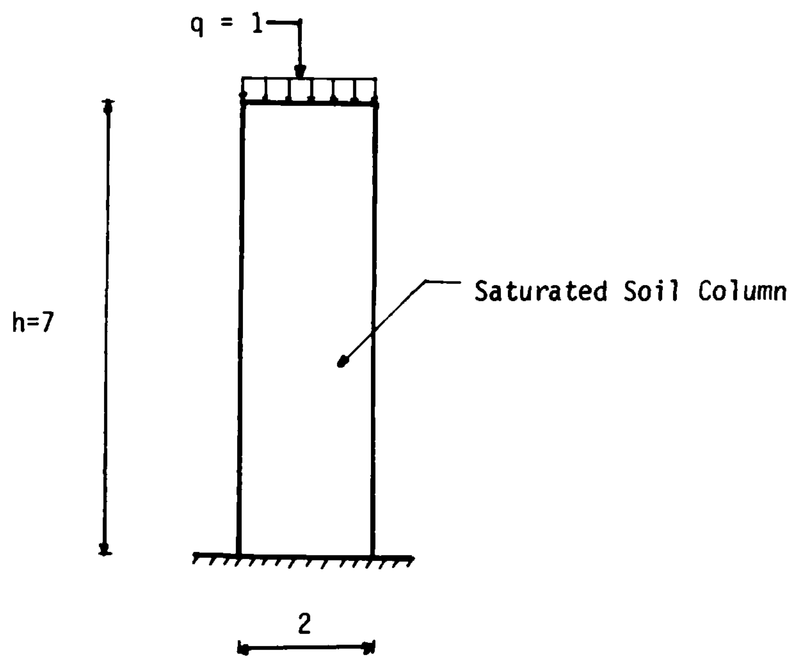


Figure 1: Terzaghi's One-Dimensional Consolidation

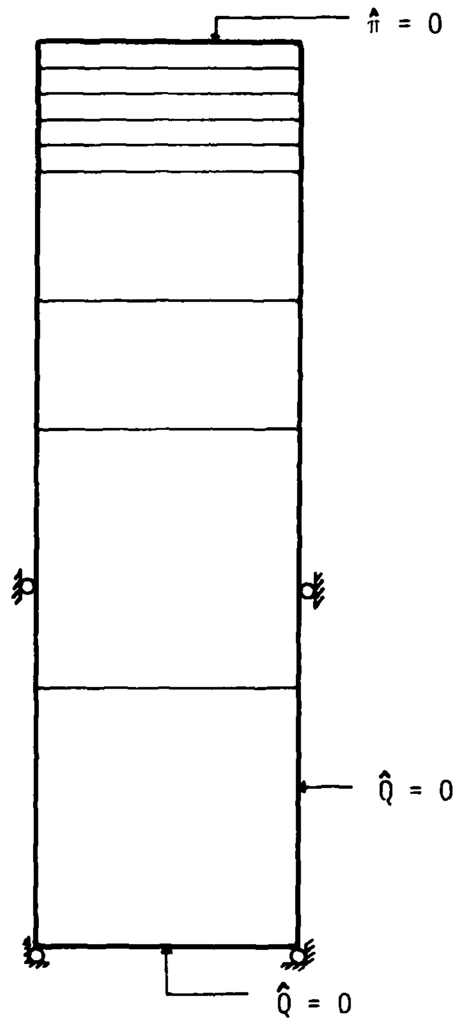


Figure 2: Finite Element Mesh for Terzaghi's Problem

The second problem was that of a saturated layer of finite thickness subjected to a strip loading. An analytical solution for this problem was developed by Gibson et al. (28). The material properties were assumed the same as for the one-dimensional problem except that Poisson's ratio was set equal to zero in the problem. The geometry and the finite element mesh are shown in Figure 3. The mesh consisted of 88 elements with 108 nodes when using the 6-4 element and 303 nodes when using the 8-4 element. Since the infinite domain had to be modeled by a finite one in the numerical schemes, the following alternative boundary conditions were specified at the far boundary where the half-space was cut off.

a. Pore fluid pressure and horizontal displacement prescribed.

b. Fluid flux and horizontal displacement prescribed.

c. Pore pressure and traction prescribed.

In temporal discretization, the factor α was set equal to 0.5, i.e. piecewise linear variation in u_i and π was assumed. The following partition was used;

10 steps of $\Delta t = 0.1$ over $[0,1]$
10 steps of $\Delta t = 1$ over $[1,11]$
10 steps of $\Delta t = 10$ over $[11,111]$
10 steps of $\Delta t = 100$ over $[111,1111]$
10 steps of $\Delta t = 1000$ over $[1111,11111]$

Here, unit is second.

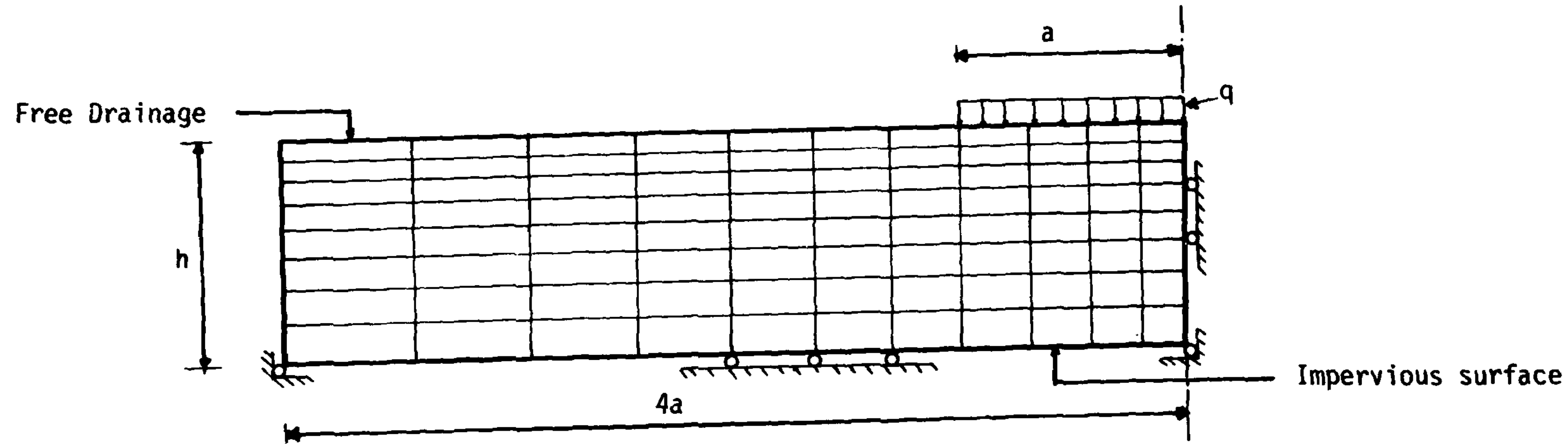


Figure 3: Finite Element Mesh for Gibson's Problem

SECTION V

RESULTS OF ANALYSIS

A. Terzaghi's Problem of One-Dimensional Cosolidation

In analyzing Terzaghi's problem, Figure 1, CPU time on the AMDAHL 470/V8 Computer was 13.92 seconds for 8-4 element while for the 6-4 element it was 10.03. Table 1 shows time settlement history for the two types of elements as well as the analytical solution. The response of the two elements practically coincided throughout the time domain and was in good agreement with the analytical solution, except at early time i.e. $t < 0.1$. The pore-pressure distribution generated by the two elements for different time steps is shown in Figure 4. The vertical axis shows the ratio of depth below surface to the total height h of the soil column considered. The horizontal axis gives the pore fluid pressure as a fraction of the intensity of the constant surface load. The plots are for values of time variable equal to 0.000001, 11.1, 201.1, and 901.1 corresponding to non-dimensional time factor $\tau = \bar{C}t/h^2$ of $0.1\text{x}10^{-8}$, 0.01165, 0.2110, 0.9195 where $\bar{C} = 2GK$ and K is the coefficient of permeability. The plots of pressure distribution history generated by the two schemes coincide. At early stages, the error in the pore pressure at points near the loaded surface is quite large for both the schemes. This is a feature of the spatial interpolation (8) used.

TABLE 1

Surface Settlement History

Time (sec)	8-4 Element	6-4 Element	Exact(Ref.23)
0.2	0.51917×10^{-5}	0.51583×10^{-5}	0.28147×10^{-5}
0.6	0.15881×10^{-5}	0.15849×10^{-5}	0.15417×10^{-5}
1.1	0.21221×10^{-4}	0.21189×10^{-4}	0.20874×10^{-4}
21.1	0.91394×10^{-4}	0.91365×10^{-4}	0.91423×10^{-4}
41.1	0.12815×10^{-3}	0.12813×10^{-3}	0.12760×10^{-3}
81.1	0.18022×10^{-3}	0.18019×10^{-3}	0.17924×10^{-3}
301.1	0.34446×10^{-3}	0.34450×10^{-3}	0.34205×10^{-3}
901.1	0.50352×10^{-3}	0.50351×10^{-3}	0.50166×10^{-3}

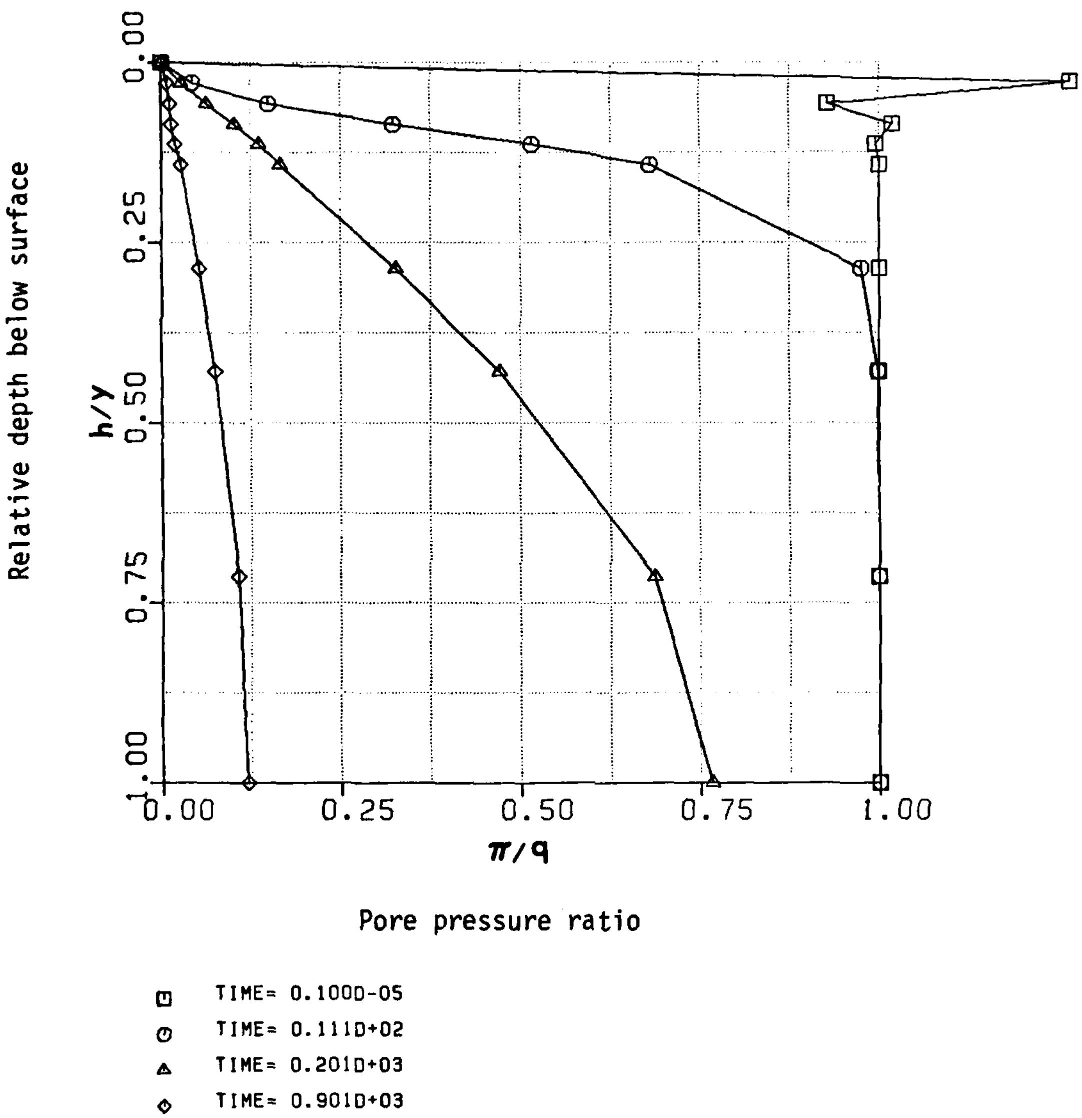


Figure 4: Fluid Pressure Distribution at Various Time Stages

B. Gibson's Problem of Plane Strain Consolidation under a Strip Load

In analyzing the second problem, the CPU time on the AMDAHL 470/V8 Computer for 8-4 element was 315 seconds while for the 6-4 element it was 33 seconds. Table 2 shows vertical displacement and pore pressure at the distance 0.084a below the surface directly under the center of the loaded area. Here, a is half-width of the loaded strip. A non-dimensional measure of the settlement is introduced as Gu/aq where G is the shear modulus, q the intensity of load per unit width of the strip. The fluid pressure is represented by the fraction π/q. The quantities are tabulated for dimensionless time given by $\tau = \bar{C}t/h^2$. Good agreement between the results for the two finite element procedures is seen at all time stages. Table 3 and Figure 5 show comparison between the settlement and pore pressure history for cases a and b, using the 6-4 element under different conditions at the "cut off" boundary. From Table 3 and Figure 5, it appears that the location of the "far" boundary was selected far enough so that, for the case where horizontal displacements are assumed to vanish at this distace, prescribing vanishing fluid pressure or fluid velocity at this boundary had little effect on the displacements and fluid pressures near the center of the loaded area.

Comparing case c with case a, i.e. considering the effect of traction or displacements at the "far" boundary for prescribed fluid pressure, it was found, Figure 6, that the settlement in the early stage of loading was significantly affected. The solution for case c was in excellent agreement with Gibson's (28) theoretical solution.

TABLE 2

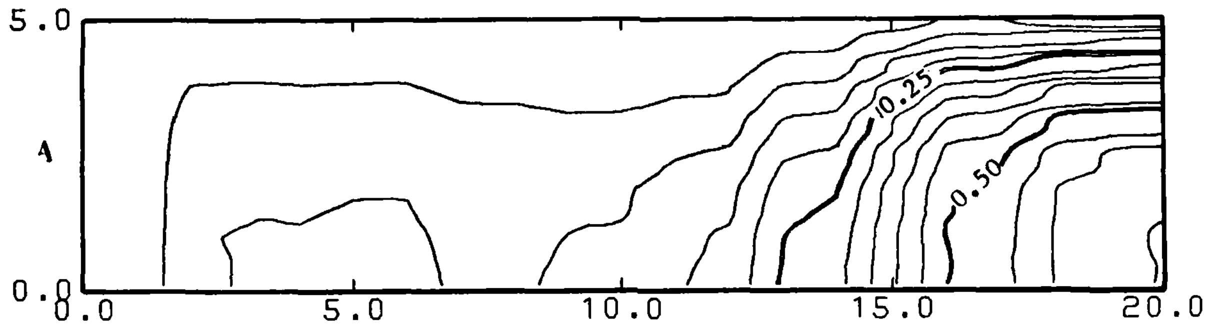
Vertical Displacement and Pore Pressure at 0.084a below the Center of the Loaded Area. Case a: Fluid Pressure Prescribed at the "far" Boundary

$\tau = \frac{\bar{C}t}{h^2}$	Gu/aq		π/q	
	8-4	6-4	8-4	6-4
9.6×10^{-5}	0.176136	0.177012	0.88874	0.9002
3.84×10^{-4}	0.180666	0.181698	0.48602	0.48610
0.01056	0.190212	0.191256	0.32230	0.32164
0.03936	0.218784	0.22014	0.17829	0.18030
0.10656	0.262584	0.264534	0.11498	0.11526
0.39456	0.365376	0.36753	0.50406	0.50271
0.68256	0.413892	0.416008	0.25815	0.25778
1.06656	0.443682	0.445914	0.01117	0.01117
3.94656	0.466842	0.469134	0.1174x10	0.115x10

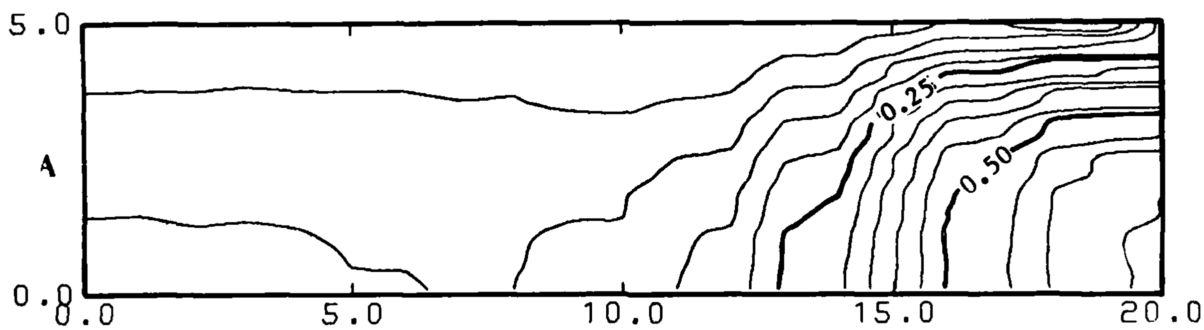
TABLE 3

Comparison of Case a and b: Vertical Displacement and Pore Pressure at 0.084a below the Center of the Loaded Area

$\tau = \frac{\bar{C}t}{h^2}$	Gu/aq		π/q	
	Case a	Case b	Case a	Case b
3.6×10^{-5}	0.17855	0.177012	0.89619	0.90026
9.6×10^{-4}	0.17926	0.17769	0.73221	0.73562
0.01056	0.192738	0.191256	0.32014	0.32164
0.04896	0.229296	0.22779	0.16341	0.16434
0.10656	0.265962	0.264534	0.11456	0.11526
0.87456	0.433548	0.434202	0.017178	0.016863
1.06656	0.445332	0.445914	0.011439	0.011163



(a)



(b)

Figure 5: Pore Pressure Contours by 6-4 Element at $\bar{C}t/h^2$=0.5856 (a) Free drainage is permitted along A. (b) No drainage along A.

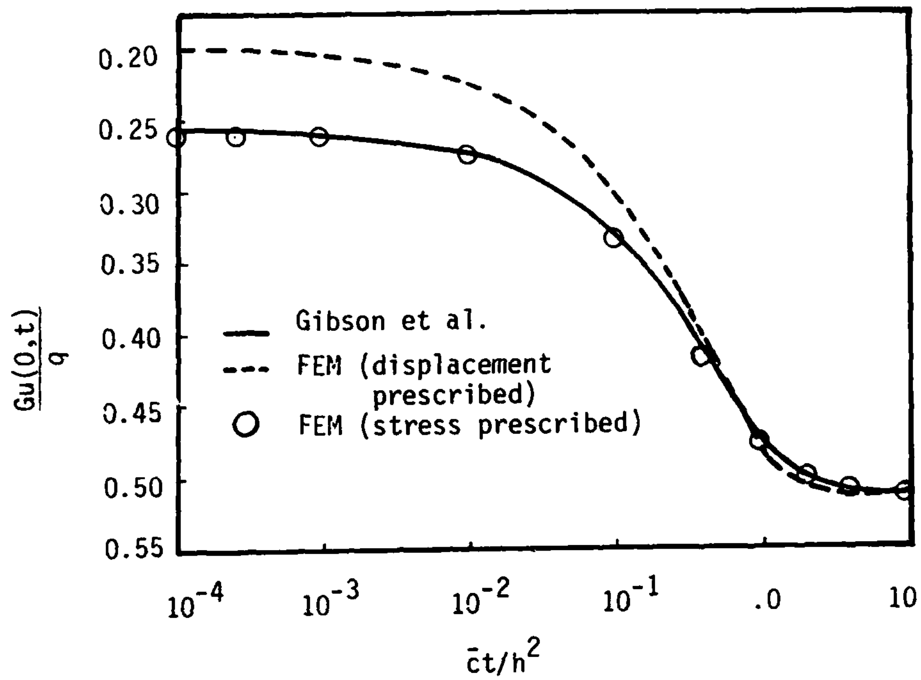


Figure 6: Influence of "Far" Boundary Conditions on Surface Settlement History

SECTION VI

CONCLUSIONS

Ghaboussi and Wilson's 6-4 element and Sandhu's 8-4 element were applied to Terzaghi's and Gibson's problem for which exact solutions are available. Results of these limited tests show;

i. The 6-4 element gave a solution identical to that given by the 8-4 element but with significant savings in computational time.

ii. The 6-4 element as well as the 8-4 element predicted the undrained solution ($t = 0+$).

iii. At early stage of loading, both elements gave unsatisfactory results. Apparently, special singularity elements (29) are required near loaded drained surfaces.

iv. The conditions at the "cut off" boundary for the finite domain considered in the finite element model must be carefully defined. The solution near the loaded region was not sensitive to whether the fluid flux or fluid presure was prescribed. However, prescribed displacement or tractions had significant influence. This is in line with Sandhu and Schiffman's experience (30).

v. The 6-4 element is distinctly superior to the 4-4 element in that it gives the solution at time = 0+ and does not have the oscillatory error of the 4-4 element. At the same time, it has the economy of the simpler element.

vi. Using the 8-4 element to generate the solution at time t = 0+, numbering of the nodal points has to be carefully ordered to avoid zeroes on the diagonal of the matrix. However, Ghaboussi and Wilson's 6-4 element is free from this defect. This is because, in eliminating the additional degrees of freedom, the static condensation would, in general, result in non-zero diagonal quantities.

vii. Finally, it appears that Ghaboussi and Wilson's 6-4 element is a good candidate for consideration towards application in nonlinear and dynamic problems as well as for extension to three-dimensional applications.

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APPENDIX A

EQUATIONS OF CONSOLIDATION

In this appendix, we list the field equations governing the quasistatic deformation of soil-water mixtures, following Biot's theory (31).

A.1 PRELIMINARIES

Let R be the open connected region occupied by the fluid-solid mixture. Let ∂R be the boundary of R and $\bar{R}$ its closure. The domain of definitions of all functions of interest is the cartesian product R x $[0,\infty)$. Here, $[0,\infty)$ denotes the positive interval of time.

Let $u_i(\underline{x},t)$, $U_i(\underline{x},t)$, $w_i(\underline{x},t)$, $e_{ij}(\underline{x},t)$, $\sigma_{ij}(\underline{x},t)$, $\tau_{ij}(\underline{x},t)$, $f_i(\underline{x},t)$, $q_i(\underline{x},t)$ and $\theta_i(\underline{x},t)$, in that order, denote cartesian components of the displacement vector, the fluid displacement vector, the relative displacement of the fluid with respect to the solid, the infinitesimal strain tensor, the symmetric Cauchy effective stress tensor in the soil, the total stress tensor, the body force vector, the fluid flux vector and the quantity, conjugate to the fluid flux vector, denoted by vector $\underline{\theta}$. Also, let $\pi(\underline{x},t)$ denote the isotropic fluid stress above a reference state (throughout, we shall consider the atmospheric pressure to be the reference state). Let ρ denote the bulk density of the mixture. ρ equals the sum of the equivalent bulk density ρ_1 of the soil and ρ_2 of the fluid. Following Biot (31), we assume the stresses $\underline{\sigma}(\underline{x},t)$ and $\pi(\underline{x},t)$ to act, respectively, over the solid and the fluid areas of any surface element.

A.2 THE FIELD EQUATIONS

A.2.1 Kinematics

For small strains, infinitesimal strain tensor is expressed in terms of displacement components as

$$e_{ij} = 1/2\,(u_{i,j} + u_{j,i}) = u_{(i,j)} \tag{A-1}$$

Displacement of the fluid relative to the solid but measured in terms of volume per unit area of the bulk medium is defined by

$$w_i \equiv f(\dot{U}_i - \dot{u}_i) \tag{A-2}$$

where f is the porosity. Accordingly, for uniform porosity, the fluid relative volumetric strain is given by

$$\xi = f(U_{i,i} - u_{i,i}) \tag{A-3}$$

A.2.2 Constitutive Equations

The constitutive equations have to be written for the stresses as well as for diffusive resistance.

A.2.2.1 Total and Pore Pressure

Biot (31), assuming the existence of a strain energy function for the mixture, postulated the following constitutive relations for compressible fluid following through linearly elastic solid.

$$\tau_{ij} = E_{ijkl}e_{kl} + \alpha\pi\delta_{ij} \quad \text{on } \overline{R} \times [0,\infty) \tag{A-4}$$

$$\pi = M\ (\alpha\, e_{kk} + \zeta) \qquad \text{on } \bar{R} \times [0,\infty) \tag{A-5}$$

where E_{ijkl}, α, M are material coefficients.

A.2.2.2 Diffusive Resistance

Diffusive resistance in a binary mixture of soil and water, for no inertia effects, is defined (1) by the components

$$\pi_i \equiv -\pi_{,i} - \rho_2 f_i \tag{A-6}$$

$$\equiv \sigma_{ji,j} + \rho_1 f_i \tag{A-7}$$

For the linear theory, assuming irrotational isothermal flow, the diffusive resistance is linearly proportional to the relative velocity of the fluid-solid constituents. This leads to

$$\pi_{,i} + \rho_2 f_i = \theta_i = -\pi_i = C_{ij} w_j \tag{A-8}$$

where w_j are components of the nominal relative velocity and C_{ij} are are components of the flow-resistivity tensor. Symmetry requirements imply

$$C_{ij} = C_{ji} \tag{A-9}$$

If the resistivity tensor has an inverse denoted by components K_{ij}, Equation (A-8) gives a generalization of Darcy law

$$w_i = q_i = K_{ji}\theta_j \tag{A-10}$$

The permeability tensor is symmetric so that

$$K_{ij} = K_{ji} \tag{A-11}$$

A.3 BALANCE LAWS

A.3.1 Force Equilibrium

Balance of forces on infinitesimal volumes, assuming small deformation, neglecting inertia and in the absence of body couples is expressed by equations

$$\left.\begin{aligned} &\tau_{ij,j} + \rho f_i = 0 \qquad \text{(A-12)} \\ &\tau_{ij} = \tau_{ji} \qquad \text{(A-13)} \end{aligned}\right\} \quad \text{on } \bar{R} \times [0,\infty)$$

A.3.2 Mass Continuity

For no chemical reaction between constituents of a binary mixture, the mass of each within a fixed volume must be conserved. Referring to material frame associated with the soil skeleton, for compressible fluid, continuity impies continued saturation of the soil. Formally, we write

$$\dot{\xi} = K_{ij}(\pi_{,j} + \rho_2 f_j)_{,i} \tag{A-14}$$

Combining Equations (A-5) and (A-14) to eliminate ξ,

$$-\alpha\, \dot{e}_{ii} + (1/M)\dot{\pi} = K_{ij}(\pi_{,j} + \rho_2 f_j)_{,i} \tag{A-15}$$

i.e., for compressible fluid and solid, the rate of the outflow from a material volume of soil skeleton equals the rate of decrease of the volume. Equation (A-12) upon substitution of Equation (A-4) and (A-2) yields Equation (1) of SECTION II. Equation (2) in that section is a rearrangement of Equation (A-15).

APPENDIX B

GHABOUSSI AND WILSON'S ELEMENT

In this appendix, we outline the development of matrices for the 6-4 element.

B.1 GEOMETRY

The cartesian coordinates of any arbitrary point in a quadrilateral may be written as

$$x = \sum_{i=1}^{4} x_i N_i \tag{B-1}$$

$$y = \sum_{i=1}^{4} y_i N_i \tag{B-2}$$

where N_i are the interpolating functions and x_i, y_i are the coordinates of the four nodal points of the quadrilateral.

B.2 DISPLACEMENT

For an isoparametric element, the interpolating functions for the displacement and geometry are the same. However, in this element two extra modes are used in defining displacements, i.e.

$$u = \sum_{i=1}^{6} u_i N_i \tag{B-3}$$

$$v = \sum_{i=1}^{6} v_i N_i \tag{B-4}$$

where u and v are the components of displacement vector along x and y direction. u_i, v_i are the nodal point displacements or the generalized coordinates associated with the additional modes.

B.3 PRESSURE

The fluid pressure at an arbitrary point of the isoparametric element is

$$\pi = \sum_{i=1}^{4} \pi_i N_i = \underline{N}_\pi \underline{\pi} \qquad \text{(B-5)}$$

where $\underline{\pi}$ is the vector of nodal pressure. In Equations (B-1) through (B-5), the N_i, i=1,6 are

$$\begin{aligned}
N_1 &= 1/4\ (1-s)(1-t) \\
N_2 &= 1/4\ (1+s)(1-t) \\
N_3 &= 1/4\ (1+s)(1+t) \\
N_4 &= 1/4\ (1-s)(1+t) \\
N_5 &= (1-s^2) \\
N_6 &= (1-t^2)
\end{aligned} \qquad \text{(B-6)}$$

where s, t are the local coordinates of the element such that the opposite pairs of sides of the element are represented by $s = \pm 1$ and $t = \pm 1$.

B.4 STRAIN-DISPLACEMENT RELATIONSHIP

$$\begin{Bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ \\ v \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} \sum_{i=1}^{6} u_i N_i \\ \\ \sum_{i=1}^{6} v_i N_i \end{Bmatrix}$$

$$= \begin{Bmatrix} \sum_{i=1}^{6} N_{i,x} u_i \\ \sum_{i=1}^{6} N_{i,y} v_i \\ \sum_{i=1}^{6} (N_{i,x} v_i + N_{i,y} u_i) \end{Bmatrix} = \underset{\sim}{B}_e \, \underset{\sim}{U} \tag{B-7}$$

where $\underset{\sim}{U}$ is the vector of nodal and generalized displacement and $\underset{\sim}{B}_e$ is an appropriate transformation. The volumetric strain is given by

$$\Delta = e_{xx} + e_{yy} = \sum_{i=1}^{6} N_{i,x} u_i + \sum_{i=1}^{6} N_{i,y} v_i = \underset{\sim}{B}_\Delta^T \, \underset{\sim}{U} \tag{B-8}$$

B.5 PRESSURE GRADIENT

Components of the pressure gradient are;

$$\begin{Bmatrix} \theta_x \\ \\ \theta_y \end{Bmatrix} = \begin{Bmatrix} \pi_{,x} \\ \\ \pi_{,y} \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{4} N_{i,x} \pi_i \\ \\ \sum_{i=1}^{4} N_{i,y} \pi_i \end{Bmatrix} = \underset{\sim}{B}_q \pi \tag{B-9}$$

B.6 EVALUATION OF DERIVATIVES $N_{i,x}$ AND $N_{i,y}$

Equation (B-6) gives the interpolating functions in terms of the local coordinates s,t. To evaluate derivatives with respect to global coordinates x,y, we use the relationship

$$\begin{Bmatrix} N_{i,x} \\ N_{i,y} \end{Bmatrix} = 1/J \begin{bmatrix} \partial y/\partial t & -\partial y/\partial s \\ -\partial x/\partial t & \partial x/\partial s \end{bmatrix} \begin{Bmatrix} N_{i,s} \\ N_{i,t} \end{Bmatrix} \tag{B-10}$$

where $J=(\sum_{i=1}^{4}N_{i,s}x_i)(\sum_{i=1}^{4}N_{i,t}y_i) - (\sum_{i=1}^{4}N_{i,s}y_i)(\sum_{i=1}^{4}N_{i,t}x_i)$. Equation (B-10) is obtained by inverting the expressions of $N_{i,s}$ and $N_{i,t}$ in terms of $N_{i,x}$ and $N_{i,y}$ by the chain rule of differentiation.

B.7 ELEMENT MATRICES

The element matrices contributing to the system described by Equation (7) are defined as follows.

$$K_{uu} = \int_{-1}^{1}\int_{-1}^{1} \underset{\sim}{B}_e^T \, \underset{\sim}{D} \, \underset{\sim}{B}_e \, J \, ds dt \tag{B-11}$$

$$K_{pp} = \int_{-1}^{1}\int_{-1}^{1} \underset{\sim}{B}_q^T \, \underset{\sim}{k} \, \underset{\sim}{B}_q \, J \, ds dt \tag{B-12}$$

$$K_{up} = \int_{-1}^{1}\int_{-1}^{1} \underset{\sim}{B}_\Delta \underset{\sim}{N}_\pi^T \, J \, ds dt = K_{pu}^T \tag{B-13}$$

$$C_{pp} = \int_{-1}^{1}\int_{-1}^{1} \frac{1}{M} \underset{\sim}{N}_\pi \, \underset{\sim}{N}_\pi^T \, J \, ds dt \tag{B-14}$$

Here $\underset{\sim}{D}$, $\underset{\sim}{k}$ are matrices describing the elastic properties and the permeability, respectively, of the porous material.

B.8 LOAD VECTORS

The vector $\{P_1\}$, $\{P_2\}$ on the right hand side of Equation (7) are given by

$$\{P_1\} = \{M_1\} + \{M_3\} \tag{B-15}$$

$$\{P_2\} = \alpha \Delta t \ M_2(t_1) + (1-\alpha)\ \Delta t M_2(t_o) \ - \alpha \Delta t \ M_4(t_1) \\ -(1-\alpha)\ \Delta t \ M_4(t_o) \tag{B-16}$$

where

$$\{M_1\} = \int_{-1}^{1}\int_{-1}^{1} \underset{\sim}{N}_u \ \{\rho f\} \ J \ dsdt \tag{B-17}$$

$$\{M_2\} = \int_{-1}^{1}\int_{-1}^{1} \underset{\sim}{N}_q \ k \ \{\rho f\} \ J \ dsdt \tag{B-18}$$

$$\{M_3\} = \int_S \underset{\sim}{N}_u \ \underset{\sim}{N}_t^T \ \{t\} \ dS_{2i} \tag{B-19}$$

$$\{M_4\} = \int_S \underset{\sim}{N}_\pi \ \underset{\sim}{N}_q^T \ \{Q\} \ dS_4 \tag{B-20}$$

where $\underset{\sim}{N}_u$, $\underset{\sim}{N}_\pi$ are defined in Equations (B-3), (B-4) and (B-5), and f is assumed to be constant over each element. $\underset{\sim}{N}_t$, $\underset{\sim}{N}_q$, $\underset{\sim}{N}_u$, $\underset{\sim}{N}_\pi$ represent interpolating functions for boundary tractions, boundary flux, displacements of the solid and fluid pressures, respectively. The interpolating functions $\underset{\sim}{N}_u$, $\underset{\sim}{N}_\pi$ for the consolidating region and for its boundary will be different.

B.9 ELIMINATION OF LOCAL MODES

For the 6-4 element, the discretized equations can be written for a time step (t_o, t_1) in the form,

$$\begin{bmatrix} [K_{uu}] & [K_{pu}] \\ [K_{pu}]^T & -[\alpha \Delta t[K_{pp}]-[C_{pp}] \end{bmatrix} \begin{Bmatrix} u(t_1) \\ \pi(t_1) \end{Bmatrix} = \begin{Bmatrix} R_u \\ R_\pi \end{Bmatrix} \qquad \text{(B-21)}$$

where $\{R_u\} = \{M_1\} + \{M_3\}$ (B-22)

and $\{R_\pi\} = [K_{pu}]^T\{u(t_o)\} + [(1-\alpha)\Delta t[K_{pp}]-[C_{pp}]]\{\pi(t_o)\} + \alpha\Delta t\{M_2(t_1)\}$

$$+ (1-\alpha)\Delta t\{M_2(t_o)\} - \alpha\Delta t\{M_4(t_1)\} - (1-\alpha)\Delta t\{M_4(t_o)\} \qquad \text{(B-23)}$$

To eliminate the four degrees of freedom associated with the two incompatible modes for each displacement component, the consolidation Equation (B-21) can be partitioned as;

$$\begin{bmatrix} [K_{uu11}] & [K_{uu12}] & [K_{pu1}] \\ [K_{uu21}] & [K_{uu22}] & [K_{pu2}] \\ [K_{pu1}]^T & [K_{pu2}]^T & -(\alpha\Delta t[K_{pp}]+[C_{pp}]) \end{bmatrix} \begin{Bmatrix} u_1(t_1) \\ u_2(t_1) \\ \pi(t_1) \end{Bmatrix} = \begin{Bmatrix} R_{u1} \\ R_{u2} \\ R_\pi \end{Bmatrix} \qquad \text{(B-24)}$$

where the internal degrees of freedom are denoted by u_2. Solving Equation $(B\text{-}24)_2$ for u_2,

$$u_2 = K_{uu22}^{-1}\{R_{u2} - K_{uu21}u_1 - K_{pu2}\pi\} \qquad \text{(B-25)}$$

Substituting for u_2 in Equation $(B\text{-}24)_1$ and collecting terms,

$$[K_{uu11} - K_{uu12}\, K_{uu22}^{-1}\, K_{uu21}]\, \{u_1(t_1)\}$$

$$+ [K_{pu1} - K_{uu12}\, K_{uu22}^{-1}\, K_{pu2}]\, \{\pi(t_1)\}$$

$$= \{R_{u1} - K_{uu12}\, K_{uu22}^{-1}\, R_{u2}\} \tag{B-26}$$

Similarly, substituting for u_2 in Equation $(B\text{-}24)_3$ and collecting terms,

$$[K_{pu1}^{T} - K_{pu2}^{T}\, K_{uu22}^{-1}\, K_{uu21}]\, \{u_1(t_1)\}$$

$$- [\alpha \Delta t K_{pp} + C_{pp} + K_{pu2}^{T}\, K_{uu22}^{-1}\, K_{pu2}]\, \{\pi(t_1)\}$$

$$= \{R_{\pi} - K_{pu2}^{T}\, K_{uu22}^{-1}\, R_{u2}\} \tag{B-27}$$

Equations (B-26) and (B-27) can be rewritten as

$$\begin{bmatrix} K^{*}_{uu} & K^{*}_{pu} \\ K^{*}_{pu} & K^{*}_{pp} \end{bmatrix} \begin{Bmatrix} u^{*}(t_1) \\ \pi(t_1) \end{Bmatrix} = \begin{Bmatrix} R^{*}_{u} \\ R^{*}_{\pi} \end{Bmatrix} \tag{B-28}$$

where

$$[K^{*}_{uu}] = K_{uu11} - K_{uu12}\, K_{uu22}^{-1}\, K_{uu21} \tag{B-29}$$

$$[K^{*}_{pu}] = K_{pu} - K_{uu12}^{T}\, K_{uu22}^{-1}\, K_{pu2} \tag{B-30}$$

$$[K^{*}_{pp}] = -\alpha \Delta t K_{pp} - K_{pu2}\, K_{uu22}^{-1}\, K_{pu2} - C_{pp} \tag{B-31}$$

$$\{R^{*}_{u}\} = R_{u1} - K_{uu12}\, K_{uu22}^{-1}\, R_{u2} \tag{B-32}$$

$$\{R^{*}_{\pi}\} = R_{\pi} - K_{pu2}^{T}\, K_{uu22}^{-1}\, R_{u2} \tag{B-33}$$

It is worth noting that even if the fluid is incompressible. i.e. C_{pp} is null, at $\Delta t=0$, K^*_{pp} is, in general, nonzero. This makes solution of the undrained problem possible.

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